Case 2 :- Double root of the indicial equation.

the indicial equation $r^2+(a_0-1)r+b_0=0$ has double root "r" if, and only if $(a_0-1)^2-4b=0$ and then $r=(1-a_0)/2$.

We have $y_1(x)=x^r(c_0+c_1x+c_2x^2+\cdots)$

$r=(1-a_0)/2$.

we apply the method of variation of parameters, that is, we replace the constant C in the solution y, by a function $u(x)$ to be determined. so that.

$$y_2(x)=u(x)\cdot y_1(x) \quad \text{----} \quad (*)$$

By inserting equation $(*)$ and the derivatives, we have :-

$$x^2y''+x\,a(x)\,y'+b(x)\,y=0$$

$$\therefore\ y_2'=uy_1'+u'y_1\ ,\quad y_2''=uy_1''+y_1'u'+u'y_1'+y_1u''$$

$$\therefore\ y_2''=uy_1''+2u'y_1'+u''y_1$$

$$\therefore\ x^2(u''y_1+2u'y_1'+uy_1'')+xa(u'y+uy_1')+buy_1=0.$$

Since y_1 is a solution of $x^2 y'' + x a(x) y' + b(x) y = 0$

The sum of the terms involving u is zero, and the last equation reduce to.

$$x^2 y_1 u'' + 2x^2 y_1' u' + x a y_1 u' = 0$$

divided by $x^2 y_1$ and inserting the power series for "a", we obtain :-

$$u'' + \left(2\frac{y_1'}{y_1} + \frac{a(x)}{x}\right) u' = 0$$

$$u'' + \left(2\frac{y_1'}{y_1} + \frac{a_0}{x} + \frac{a_1}{x}x + \frac{a_2}{x}x^2 + \cdots\right) u' = 0$$

$$y_1 = \sum_{m=0}^{\infty} C_m x^{r+m} = C_0 x^r + C_1 x^{r+1} + \cdots$$

$$y_1' = \sum_{m=0}^{\infty} (r+m) C_m x^{r+m-1}$$

$$\therefore \frac{y_1'}{y} = \frac{1}{x}\left[\frac{rC_0 + rC_1 x + rC_2 x^2 + \cdots}{C_0 + C_1 x + C_2 x^2 + \cdots}\right]$$

$$\therefore \frac{y_1'}{y} = \frac{1}{x}\left[\frac{r[C_0 + C_1 x + C_2 x^2 + \cdots}{C_0 + C_1 x + C_2 x^2} + \frac{C_1 + 2C_2 x^2 + \cdots}{C_0 + C_1 x + C_2 x^2}\right]$$

$$\therefore \frac{y'}{y} = \frac{r}{x} + \cdots$$

Hence the last equation can be written

$$u'' + \left[2\left(\frac{r}{x} + \cdots\right) + \frac{a(x)}{x}\right]u' = 0$$

$$u'' + \left(\frac{2r+a}{x} + \cdots\right)u' = 0$$

Since $r(1-a_0)/2$ ∴ the term $(2r+a_0)/x$ equal $\frac{1}{x}$ and divided by u' we thus have

$$\frac{u''}{u'} = -\frac{1}{x} + \cdots$$

by integration, we obtain

$$\ln u' = -\ln x + \cdots \Longrightarrow u' = \frac{1}{x} \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\therefore u = \ln x + \cdots$$

$$\therefore u = \ln x + K_1 x + K_2 x^2 + \cdots$$

$$\therefore y_2(x) = y_1(x)\ln x + x^r \sum_{m=1}^{\infty} A_m x^m$$

Ex Solve the differential equation by "Frobenius method"

$$x(x-1)y'' + (3x-1)y' + y = 0$$

We have

$$y(x) = x^r \sum_{m=0}^{\infty} C_m x^m, \quad y'(x) = (r+m)\sum_{m=0}^{\infty} C_m x^{r+m-1}$$

$$y'' = (r+m)(r+m-1)\sum_{m=0}^{\infty} C_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r-1}$$

$$+3\sum_{m=0}^{\infty}(m+r)C_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)C_m x^{m+r-1}$$

$$+\sum_{m=0}^{\infty} C_m x^{m+r} = 0$$

$$[-r(r-1)-r]C_0 = 0 \quad \text{or} \quad r^2 = 0$$

$$\therefore S(S-1)C_s - (S+1)SC_{s+1} + 3SC_s - (S+1)C_{s+1} + C_s = 0$$

$$C_{s+1} = C_s$$

$\vdots$

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

$$y_2(x) = u(x)\cdot y_1(x)$$

$$x(x-1)\left[u''y_1+2u'y_1'+uy_1''\right]+(3x-1)(u'y_1+uy_1')+uy_1=0$$

$$x(x-1)\left[u''y_1+2u'y_1'\right]+(3x-1)u'y_1=0$$

$$\vdots$$

$$\frac{u''}{u'}=-\frac{1}{x}$$

$$\therefore\ y_2=uy_1=\frac{\ln x}{1-x}.$$

<u>Case 3</u> Roots of the idicial equation differing by an integer r_1 and r_2. (equal but difference in sign)

$$y_1(x)=x^{r_1}\left[C_0+C_1x+C_2x^2+\cdots\right]$$

$$y_2(x)=Kp\,y_1(x)\ln x+x^{r_2}\sum_{m=0}^{\infty}C_m x^m$$

<u>EX</u> Solve the differential equation by Frobenien's method.

$$(x^2-1)x^2y''-(x^2+1)xy'+(x^2+1)y=0$$

$$y=\sum_{m=0}^{\infty}C_m x^{m+r},\quad y'=\sum_{m=0}^{\infty}(m+r)C_m x^{m+r-1}$$

$$y''=\sum_{m=0}^{\infty}C_m(m+r)(m+r-1)x^{m+r-2}$$

$$(x^2-1)x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r-2}-(x^2+1)x\sum_{m=0}^{\infty}(m+r)C_m x^{m+r-1}+(x^2+1)\sum_{m=0}^{\infty}C_m x^{m+r}=0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{r+m+2} - \sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{r+m}$$

$$-\sum_{m=0}^{\infty}(m+r)C_m x^{m+r+2} - \sum_{m=0}^{\infty}(m+r)C_m x^{m+r} + \sum_{m=0}^{\infty}C_m x^{m+r+2}$$

$$+\sum_{m=0}^{\infty}C_m x^{m+r} = 0$$

$$-(m+r)(m+r-1)C_m x^{r+m} - (m+r)C_m x^{m+r} + C_m x^{m+r} = 0$$

$$-r(r-1)C_0 - rC_0 + C_0 = 0$$

$$(-r^2+r)C_0 - rC_0 + C_0 = 0$$

$$-r^2 C_0 + C_0 = 0 \Rightarrow r^2 = 1 \Rightarrow r = \mp 1$$

$$\therefore r_1 = +1 \quad , r_2 = -1.$$

$$C_{s+2} = \frac{s^2}{(s+4)(s+2)} C_s \quad \text{at } r \text{ is positive}$$

$$C_1 = 0, \ C_2 = 0, \ C_3 = 0, \ C_4 = 0$$

$$\therefore y_1(x) = x^{r_1}(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots)$$

$$y_1 = x C_0$$

$$y_2(x) = K p\, y_1(x) \ln x + x^{r_2}\sum_{m=0}^{\infty} C_m x^m$$

$$y_2(x) = K x \ln x + \frac{1}{x}\sum_{m=0}^{\infty} C_m x^m$$

$$y_2'(x) = K\left(x \cdot \frac{1}{x} + \ln x\right) + \sum_{m=0}^{\infty}(m-1)C_m x^{m-2}$$

$$y_2''(x) = K\left(\frac{1}{x}\right) + \sum_{m=0}^{\infty} (m-1)(m-2)\, C_m\, x^{m-3}$$

$$\therefore (x^2-1)\, x^2 \left(\frac{K}{x} + \sum_{m=0}^{\infty} (m-1)(m-2)\, C_m\, x^{m-3}\right.$$

$$- (x^2+1)\, x\, \left[K \ln x + K + \sum_{m=0}^{\infty} (m-1)\, C_m\, x^{m-2}\right]$$

$$+ (x^2+1)\left[K \ln x + \sum_{m=0}^{\infty} C_m\, x^{m-1}\right] = 0$$

$$2Kx + \sum_{m=0}^{\infty} (s-3)^2\, C_{s-1}\, x^s - \sum_{m=0}^{\infty} (s+1)(s-1)\, C_{s+1}\, x^s = 0$$

$$C_{s+1} = \frac{(s-3)^2}{(s^2-1)}\, C_{s-1} \quad \text{----- (general)}$$

$$\therefore C_1 = 0,\ C_3 = 0,\ C_4 = 0,\ C_2 = (-2)^2\, C_0$$

$$\therefore -2Kx + (s-3)^2\, C_{s-1}\, x^s - (s+1)(s-1)\, C_{s+1}\, x^s = 0$$

at $s = 1 \Rightarrow C_0$

$$-2Kx + (-2)^2\, C_0\, x = 0$$

$$\therefore K = 2C_0 .$$

$$\therefore y_2(x) = 2C_0 + \ln x + \frac{1}{x}\, C_0$$

Bessel's Equation. Bessel Function $J_\nu(x)$

One of the most important ODEs in applied mathematics in "Bessel's equation."

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad \text{---(1)}$$

Where the parameter "ν" is a given number which is poisitive or Zero. Bessel's equation often appers if a problem shows cylindrical symmetry. To see the approach of this method, divide (1) by x^2 to get the standand form $y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$

we have $\quad y(x) = \sum_{m=0}^{\infty} c_m x^{m+r} \quad$ ---(2) $\cdot$ as the Frobenius's theory.

Substituting (2) and its first and second derivatives into Bessel's equation, we obtain

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) c_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) c_m x^{m+r}$$

$$+ \sum_{m=0}^{\infty} c_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0 \quad \text{-(3)}$$

We equate the sum of the coefficients of x^{s+r} to zero. Note that this power x^{s+r} corresponds to $m=s$ in the first, second, and fourth series, and to $m = s-2$ in the third series.

at $s=0 \Rightarrow r(r-1)C_0 + rC_0 - \nu^2 C_0 = 0$ ——— (3a)

at $s=1 \Rightarrow (r+1)rC_1 + (r+1)C_1 - \nu^2 C_1 = 0$ ——— (3b)

at $s=2,3,\ldots \Rightarrow (s+r)(s+r-1)C_s + (s+r)C_s + C_{s-2} - \nu^2 C_s = 0$ ——— (3c)

From (3a), we obtain the "indicial equation" by dropping C_0.

$$(r+\nu)(r-\nu)=0 \quad \text{———} \quad (4)$$

ther roots are $r_1 = \nu \geq 0$, $r_2 = -\nu$.

~~at~~

at $r = r_1 = \nu$ sub in eq (3b)

$$(\nu+1)\nu C_1 + (\nu+1)C_1 - \nu^2 C_1 = 0$$

$$\nu^2 C_1 + \nu C_1 + \nu C_1 + C_1 - \nu^2 C_1 = 0$$

$$(2\nu+1)C_1 = 0$$

at $r=\nu$ sub in (3c)

$$(s+\nu)(s+\nu-1)C_s + (s+\nu)C_s + C_{s-2} - \nu^2 C_s = 0$$

$$s^2 C_s + \nu s C_s - \cancel{sC_s} + \nu s C_s + \cancel{\nu^2 C_s} - \cancel{\nu C_s} + \cancel{sC_s} + \cancel{\nu C_s}$$

$$+ C_{s-2} - \cancel{\nu^2 C_s} = 0$$

$$s^2 C_s + 2\nu s C_s + C_{s-2} = 0$$

$$(s+2\nu)s C_s + C_{s-2} = 0 \quad \text{———} \quad (5)$$

$C_1 = 0$, $C_3 = 0$, $C_5 = 0$, ----

Now we use even-numbered coefficients C_s with

$S = 2m$.

$$(2m + 2\nu)\, 2m\, C_{2m} + C_{2m-2} = 0$$

$$C_{2m} = -\frac{1}{2^2 m(\nu + m)} C_{2m-2}$$

at $m = 1 \Rightarrow$ $$C_2 = -\frac{C_0}{2^2(\nu+1)}$$

at $m = 2 \Rightarrow$ $$C_4 = -\frac{C_2}{2^2 2(\nu+2)} = \frac{C_0}{2^4 2!(\nu+1)(\nu+2)}$$

⋮

$$\boxed{C_{2m} = \frac{(-1)^m C_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)}}$$ in general

$m = 1, 2, \cdots$

------- (7)

Bessel functions $J_n(x)$ for integer $\nu = n$:-

Integer values of ν are denoted by n. This is standard.
For $\nu = n$, the relation (7) becomes

$$C_{2m} = \frac{(-1)^m C_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}, \quad m = 1, 2, \cdots \qquad (8)$$

$$C_0 = \frac{1}{2^n n!} \qquad (8\text{-}1)$$

because $n!(n+1)\cdots(n+m) = (n+m)!$ in eq (8)

$$C_{2m} = \frac{(-1)^m}{2^{m+n} m!(n+m)!}, \quad m = 1, 2, \cdots$$

By inserting these coefficients into (2) and remembering that $C_1 = 0$, $C_3 = 0$, $\cdots$ we obtain a particular solution of Bessel's equation is denoted by "$J_n(x)$"

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \qquad n \geq 0$$

- at $n=0$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{m} (m!)^2}$$

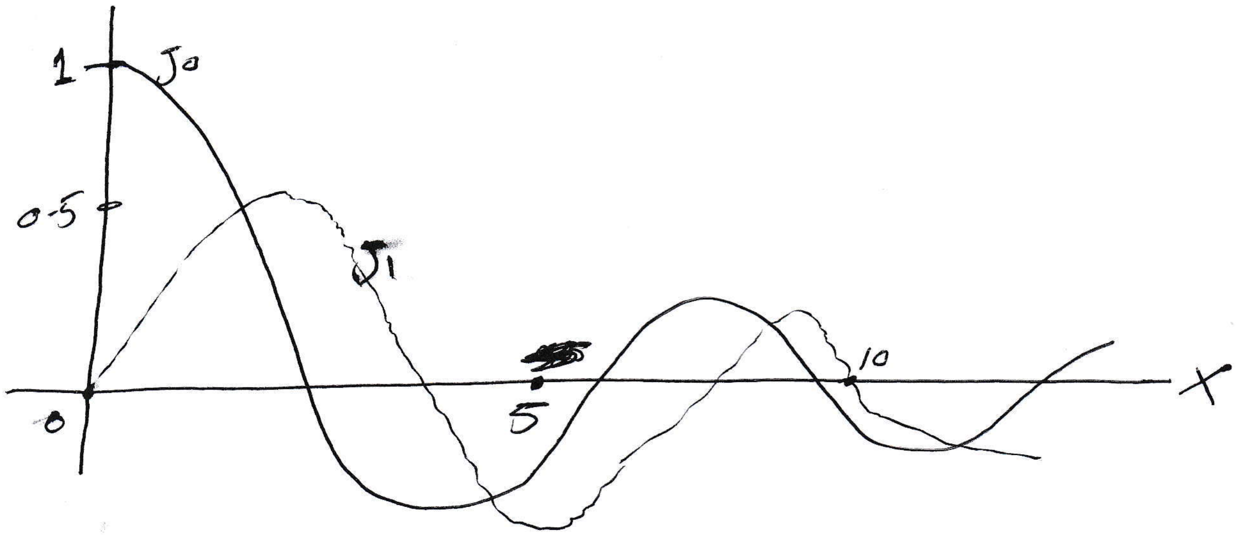
$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

- at $n=1$

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1}\, m!\, (m+1)!}$$

$$= \frac{x}{2} - \frac{x^3}{2^3\, 1!\, 2!} + \frac{x^5}{2^5\, 2!\, 3!} - \frac{x^7}{2^7\, 3!\, 4!} + \cdots$$

$$J_n(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$



Bessel function $J_\nu(x)$ for any $\nu \geq 0$. Gamma function

To extend the ~~fraction~~ factorial function $n!$ to any $\nu \geq 0$. For this we choose

$$C_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \quad \text{------- (9)}$$

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt \qquad \nu > -1 \quad \text{---- (10)}$$

Integration by parts gives

$$\Gamma(\nu+1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} \, dt = 0 + \nu \Gamma(\nu) \quad \text{---- (11)}$$

$$\therefore \Gamma(\nu+1) = \nu \Gamma(\nu). \quad \text{----- (12)}$$

at $\nu = 0 \Rightarrow \Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 0 - (-1) = 1.$

at $\nu = 1$

and $\Gamma(2) = 1\Gamma(1) = 1!$

~~and~~ at $\nu = 2$

$$\Gamma(3) = 2\Gamma(2) \Rightarrow 2!$$

$$\Gamma(n+1) = n! \quad , n = 0, 1, 2, \cdots$$

The gamma function generalizes the fractional function to arbitrary positive ν. Thus eq (9) with $\nu = n$ agrees with eq (8-1).

Furthermore, from eq (7) with C_0 given by eq (9) we first have

$$C_{2m} = \frac{(-1)^m}{2^{2m} m! (\nu+1)(\nu+2) \cdots (\nu+m) 2^{\nu} \Gamma(\nu+1)}$$

Now eq (12) gives $(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2)$,

$(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3)$ and so on,

so that

$(\nu+1)(\nu+2) \cdots (\nu+m)\Gamma(\nu+1) = \Gamma(\nu+m+1)$

Hence because of our (standard!) choice (9) a particular solution of $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$, denoted by $J_\nu(x)$ and given by

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)} \tag{12-1}$$

$J_\nu(x)$ is called the "Bessel function of the first kind of order ν".

Derivative Recursions :-

(a) $[x^{\nu} J_{\nu}(x)]' = x^{\nu} J_{\nu-1}(x)$. ---- (13-a)

(b) $[x^{-\nu} J_{\nu}(x)]' = -x^{-\nu} J_{\nu+1}(x)$. ---- (13-b)

(c) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)$. ---- (13-c)

(d) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2 J'_{\nu}(x)$. ---- (13-d)

EX To understand Bessel function

To obtain J_3,

first using (13-c) with $\nu = 2$,

$$J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$$

$$J_3(x) = 4x^{-1} J_2(x) - J_1(x)$$

then (13-c) with $\nu = 1$,

$$J_0(x) + J_2(x) = 2x^{-1} J_1(x)$$

$$J_2(x) = 2x^{-1} J_1(x) - J_0(x)$$

Substitute $J_2(x)$